Selected Solutions for Chapter 5: Probabilistic Analysis and Randomized Algorithms

Solution to Exercise 5.2-1

Since HIRE-ASSISTANT always hires candidate 1, it hires exactly once if and only if no candidates other than candidate 1 are hired. This event occurs when candidate 1 is the best candidate of the n, which occurs with probability $1/n$.

HIRE-ASSISTANT hires n times if each candidate is better than all those who were interviewed (and hired) before. This event occurs precisely when the list of ranks given to the algorithm is $\langle 1, 2, \ldots, n \rangle$, which occurs with probability $1/n!$.

Solution to Exercise 5.2-4

Another way to think of the hat-check problem is that we want to determine the expected number of fixed points in a random permutation. (A ***fixed point*** of a permutation π is a value i for which $\pi(i) = i$.) We could enumerate all $n!$ permutations, count the total number of fixed points, and divide by $n!$ to determine the average number of fixed points per permutation. This would be a painstaking process, and the answer would turn out to be 1. We can use indicator random variables, however, to arrive at the same answer much more easily.

Define a random variable X that equals the number of customers that get back their own hat, so that we want to compute $\mathrm{E}[X]$.

For $i = 1, 2, \ldots, n$, define the indicator random variable

$$X_i = \mathrm{I}\{\text{customer } i \text{ gets back his own hat}\} .$$

Then $X = X_1 + X_2 + \cdots + X_n$.

Since the ordering of hats is random, each customer has a probability of $1/n$ of getting back his or her own hat. In other words, $\Pr\{X_i = 1\} = 1/n$, which, by Lemma 5.1, implies that $\mathrm{E}[X_i] = 1/n$.

Thus,

$$
\begin{aligned}
\mathrm{E}[X] &= \mathrm{E}\left[\sum_{i=1}^{n} X_i\right] \\
&= \sum_{i=1}^{n} \mathrm{E}[X_i] \qquad \text{(linearity of expectation)} \\
&= \sum_{i=1}^{n} 1/n \\
&= 1 ,
\end{aligned}
$$

and so we expect that exactly 1 customer gets back his own hat.

Note that this is a situation in which the indicator random variables are *not* independent. For example, if $n = 2$ and $X_1 = 1$, then X_2 must also equal 1. Conversely, if $n = 2$ and $X_1 = 0$, then X_2 must also equal 0. Despite the dependence, $\Pr\{X_i = 1\} = 1/n$ for all i, and linearity of expectation holds. Thus, we can use the technique of indicator random variables even in the presence of dependence.

Solution to Exercise 5.2-5

Let X_{ij} be an indicator random variable for the event where the pair $A[i], A[j]$ for $i < j$ is inverted, i.e., $A[i] > A[j]$. More precisely, we define $X_{ij} = \mathrm{I}\{A[i] > A[j]\}$ for $1 \le i < j \le n$. We have $\Pr\{X_{ij} = 1\} = 1/2$, because given two distinct random numbers, the probability that the first is bigger than the second is $1/2$. By Lemma 5.1, $\mathrm{E}[X_{ij}] = 1/2$.

Let X be the the random variable denoting the total number of inverted pairs in the array, so that

$$
X = \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} X_{ij} .
$$

We want the expected number of inverted pairs, so we take the expectation of both sides of the above equation to obtain

$$
\mathrm{E}[X] = \mathrm{E}\left[\sum_{i=1}^{n-1} \sum_{j=i+1}^{n} X_{ij}\right] .
$$

We use linearity of expectation to get

$$
\begin{aligned}
\mathrm{E}[X] &= \mathrm{E}\left[\sum_{i=1}^{n-1} \sum_{j=i+1}^{n} X_{ij}\right] \\
&= \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \mathrm{E}[X_{ij}] \\
&= \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} 1/2
\end{aligned}
$$

$$
\begin{aligned}
&= \binom{n}{2}\frac{1}{2} \\
&= \frac{n(n-1)}{2}\cdot\frac{1}{2} \\
&= \frac{n(n-1)}{4} .
\end{aligned}
$$

Thus the expected number of inverted pairs is $n(n-1)/4$.

Solution to Exercise 5.3-2

Although PERMUTE-WITHOUT-IDENTITY will not produce the identity permutation, there are other permutations that it fails to produce. For example, consider its operation when $n = 3$, when it should be able to produce the $n! - 1 = 5$ non-identity permutations. The **for** loop iterates for $i = 1$ and $i = 2$. When $i = 1$, the call to RANDOM returns one of two possible values (either 2 or 3), and when $i = 2$, the call to RANDOM returns just one value (3). Thus, PERMUTE-WITHOUT-IDENTITY can produce only $2 \cdot 1 = 2$ possible permutations, rather than the 5 that are required.

Solution to Exercise 5.3-4

PERMUTE-BY-CYCLIC chooses *offset* as a random integer in the range $1 \le \textit{offset} \le n$, and then it performs a cyclic rotation of the array. That is, $B[((i + \textit{offset} - 1) \bmod n) + 1] = A[i]$ for $i = 1, 2, \ldots, n$. (The subtraction and addition of 1 in the index calculation is due to the 1-origin indexing. If we had used 0-origin indexing instead, the index calculation would have simplied to $B[(i + \textit{offset}) \bmod n] = A[i]$ for $i = 0, 1, \ldots, n - 1$.)

Thus, once *offset* is determined, so is the entire permutation. Since each value of *offset* occurs with probability $1/n$, each element $A[i]$ has a probability of ending up in position $B[j]$ with probability $1/n$.

This procedure does not produce a uniform random permutation, however, since it can produce only n different permutations. Thus, n permutations occur with probability $1/n$, and the remaining $n! - n$ permutations occur with probability 0.